

On the $f(R)$ theories equivalent to general relativity for spacetimes with a constant Ricci scalar

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Abstract

We prove the necessary and the sufficient condition on the $f(R)$ function involving the equivalence of GR and the $f(R)$ theory, in the sense that for a distribution of matter in a spacetime of constant Ricci scalar, GR and the $f(R)$ theories generated by the functions that satisfy this condition, will necessarily have the same solutions. We show how this condition allows to get such functions, and gives the possibility to fix conditions on the parameters of any given class of $f(R)$ functions, so that GR and this class of theories, will have exactly the same solutions. The well-known model $R + \alpha\beta^{1-p}R^p$, for dark energy is discussed as an illustration.

Key words: General relativity, $f(R)$ modified theory of gravity, Ricci scalar, common solutions, equivalence.

1 Introduction

Physical laws can be obtained from the fascinating variational principal, and general relativity theory which is by excellence the theory describing gravity,

does not make an exception to this rule, indeed Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}; \kappa = \frac{8\pi G}{c^4}, \quad (1)$$

are obtained from the variation with respect to the metric $g_{\mu\nu}$, of the Hilbert-Einstein action

$$S = \frac{1}{2\kappa} \int \sqrt{-g} (R - 2\Lambda) d^4x + S_m. \quad (2)$$

The idea of what is called $f(R)$ modified gravity, is based on the generalization of the Hilbert-Einstein action, by considering the more general action

$$S = \frac{1}{2\kappa} \int \sqrt{-g} f(R) d^4x + S_m, \quad (3)$$

which leads after the variation with respect to $g_{\mu\nu}$, to the $f(R)$ field equations

$$f'(R) R_{\mu\nu} - \frac{1}{2}f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + \square f'(R) g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (4)$$

were $f'(R) = \frac{df(R)}{dR}$ and ∇_μ represents the covariant derivative, so that for every $f(R)$ function, we have mathematically, a theory, and Einstein's one becomes the particular case were $f(R) = R - \Lambda$, among an infinity of others possibilities. Such type of modification, is one way, to go beyond Einstein's theory[1], in hope to be able to give answers, for many problems in physics, like the question of dark energy in cosmology, were $f(R)$ modified gravity is a candidate to fully understand the acceleration of the expansion of the universe, were different $f(R)$ models are presented to achieve this goal[2]. Obviously, any proposed $f(R)$ model, have to be in agreement with experiments and observations, in order to be acceptable as a viable model for dark energy, consequently many conditions must be satisfied by the $f(R)$ function.

But even if all the conditions required for the $f(R)$ function, are satisfied, it's rightful to ask the question: is it possible that GR solutions, remains solutions of field equations generating by an $f(R)$ model, when the two theories are used in the same physical context?. By a "solution" of Einstein's equations we mean a metric tensor $g_{\mu\nu}$, which is the fundamental mathematical object in the description of the geometry of the spacetime, and we mean by "the same physical context", the fact to use the same stress-energy tensor $T_{\mu\nu}$ describing a specific type of matter, for both theories. Actually there is two reasons to ask this question, the first one is due to the difficulty to get exact solutions of $f(R)$ field equations, so it becomes very useful to know if general relativity ones remains solutions of $f(R)$ theory, which allow us to bring back to Einstein's field equations, in general easier to solve than $f(R)$ ones, were most of the interesting exact solutions, from the physical point of view, are well known since long time ago[3]. The second raison is to have more information about how much a given $f(R)$ modified theory, is different from the Einstein's one.

2 The equivalence GR - $f(R)$ theory, when Ricci scalar is a constant

In order to give an answer to the previous question, we consider a spacetime of constant scalar curvature k , where the distribution of matter is described by the stress energy-tensor $T_{\mu\nu}$. We notice that it's not hard to bring up Einstein's equations, in the $f(R)$ field equations(4), which reduce when $R = k$, to

$$f'(k) R_{\mu\nu} - \frac{1}{2} f(k) g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (5)$$

and can be written in the equivalent form

$$\begin{aligned} & f'(k) \left[R_{\mu\nu} - \frac{k}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} - \kappa T_{\mu\nu} \right] \\ & + \left[\left(\frac{k}{2} - \Lambda \right) f'(k) - \frac{1}{2} f(k) \right] g_{\mu\nu} \\ & + (f'(k) - 1) \kappa T_{\mu\nu} = 0, \end{aligned} \quad (6)$$

consequently, we have the following theorem.

Theorem For any $g_{\mu\nu}$ solution of Einstein's field equations(1), such as $R = g^{\mu\nu} R_{\mu\nu} = k$, an arbitrary constant, and $f(R)$ a differentiable function, we have:

If $f'(k) \neq 0$, then $g_{\mu\nu}$ will necessarily be a solution of $f(R)$ field equations(5), if and only if

$$\left[\left(\frac{k}{2} - \Lambda \right) f'(k) - \frac{1}{2} f(k) \right] g_{\mu\nu} + (f'(k) - 1) \kappa T_{\mu\nu} = 0. \quad (7)$$

If $f'(k) = 0$, we distinguish two possibilities:

1/ $f(k) \neq \frac{k-4\Lambda}{2}$; then $g_{\mu\nu}$ will never be a solution of field equations generated by the $f(R)$ function, in other words, for a given Stress-energy tensor $T_{\mu\nu}$, it exists no common solutions between GR and the $f(R)$ theory, leading to a spacetime with a constant Ricci scalar $R = k$.

2/ $f(k) = \frac{k-4\Lambda}{2}$, then all the GR solutions, describing the geometry of a spacetime with zero Ricci tensor, and all the GR vacuum solutions, are necessarily solutions of the $f(R)$ theory.

Proof. Let $g_{\mu\nu}$ be a GR solution leading to $R = k$. To remain solution of the $f(R)$ theory, $g_{\mu\nu}$ and the $f(R)$ function must satisfy (5) when $f'(k) \neq 0$, but field equations (5) are equivalent to (6), so $g_{\mu\nu}$ must satisfy the equations (6), where the first term represents Einstein equations and consequently it vanishes, so the remaining terms are $\left[\left(\frac{k}{2} - \Lambda \right) f'(k) - \frac{1}{2} f(k) \right] g_{\mu\nu} + (f'(k) - 1) \kappa T_{\mu\nu}$, and it must vanish to allow $g_{\mu\nu}$ to be an $f(R)$ solution. The inverse is also true; indeed, let $g_{\mu\nu}$ be an $f(R)$ solution such as $R = g^{\mu\nu} R_{\mu\nu} = k$, so we have necessarily (6), then if $g_{\mu\nu}$ satisfy(7), it implies that $R_{\mu\nu} - \frac{k}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} - \kappa T_{\mu\nu} = 0$, because $f'(k) \neq 0$, therefore (7) is a necessary and a sufficient condition for the existence

of common solutions $g_{\mu\nu}$ between GR and the $f(R)$ theory, involving a constant Ricci scalar k .

If $f'(k) = 0$, then $f(R)$ field equations take the following form

$$-\frac{f(k)}{2}g_{\mu\nu} = \kappa T_{\mu\nu}. \quad (8)$$

Lets assume that $g_{\mu\nu}$ is a GR solution so that $k = g^{\mu\nu}R_{\mu\nu}$, so necessarily

$$\kappa T = 4\Lambda - k, \quad (9)$$

which is the trace of Einstein equations(1), but if $g_{\mu\nu}$ is also solution of an $f(R)$ theory were $f'(k) = 0$, and $f(k) \neq \frac{k-4\Lambda}{2}$, (8) is always true, and its trace, implies that $\kappa T \neq 4\Lambda - k$, in contradiction with (9), consequently GR solutions are completely different from $f(R)$ ones, when $f'(k) = 0$, and $f(k) \neq \frac{k-4\Lambda}{2}$.

If $f(k) = \frac{k-4\Lambda}{2}$, then $f(R)$ field equations are written

$$\frac{4\Lambda - k}{4}g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (10)$$

which shows clearly that any mertric tensor $g_{\mu\nu}$ implying $R = 4\Lambda$, will be a vacuum solution of the $f(R)$ theory, so GR vacuum solutions constitute an $f(R)$ vacuum solutions. We also deduce that for spacetimes with zero Ricci tensor, $f(R)$ field equations(10) are exactly GR ones, when $f(0) = -2\Lambda$, and $f'(0) = 0$. ■

Corollary 1 *For a given Stress-energy tensor $T_{\mu\nu}$, if $g_{\mu\nu}$ is any common solution between GR and $f(R)$ theory, such as $R = g^{\mu\nu}R_{\mu\nu} = k$, then the $f(R)$ function satisfy necessarily the condition:*

$$kf'(k) - 2f(k) + k - 4\Lambda = 0. \quad (11)$$

Proof. This condition represents the trace of (7), when we take into consideration (9), and it is the necessary and the sufficient condition so that $g_{\mu\nu}$, will be a common solution between the two theories when $R = k$. ■

Corollary 2 *For all $g_{\mu\nu}$ solution of Einstein's field equations with cosmological constant, such as $R = g^{\mu\nu}R_{\mu\nu} = k$, if*

$$\begin{aligned} f(k) &= k - 2\Lambda \\ f'(k) &= 1, \end{aligned} \quad (12)$$

then $g_{\mu\nu}$, is necessarily a solution of field equations generated by the $f(R)$ function.

Proof. We have proved that the GR solution $g_{\mu\nu}$ leading to $R = k$, will be a solution of an $f(R)$ theory, if and only if (7) is satisfied, and it's the case when $f(k) = k - 2\Lambda$ and $f'(k) = 1$. obviously the inverse is also true, i.e if $g_{\mu\nu}$ is a solution of $f(R)$ theory such as $R = k$, it's sufficient to have(12), to proof that $g_{\mu\nu}$ is also solution of GR, as we can directly see it from(6). ■

Corollary 3 *Vacuum solutions of $f(R)$ modified gravity, leading to $R = 4\Lambda$, are vacuum solutions of Einstein's field equations with cosmological constant, if and only if*

$$f'(4\Lambda) - \frac{1}{2\Lambda}f(4\Lambda) = 0, \text{ and } f'(4\Lambda) \neq 0. \quad (13)$$

Vacuum solutions of Einstein's field equations with cosmological constant, are vacuum solutions of an $f(R)$ theory, If $f'(4\Lambda) = 0$, and $f(4\Lambda) = 0$.

All vacuum solutions of Einstein's field equations without cosmological constant, are vacuum solutions of field equations, generating by any $f(R)$ function, that vanishes at zero value:

$$f(0) = 0. \quad (14)$$

Proof. Let $g_{\mu\nu}$ be a vacuum solution of Einstein's field equations(1), from (9), we have $k = 4\Lambda$, so according to (7), when $f'(4\Lambda) \neq 0$, $g_{\mu\nu}$ will necessarily be a solution of $f(R)$ field equations, if and only if

$$\left(\Lambda f'(4\Lambda) - \frac{1}{2}f(4\Lambda) \right) g_{\mu\nu} = 0, \quad (15)$$

but by hypothesis $g_{\mu\nu} \neq 0$, therefore $f'(4\Lambda) - \frac{1}{2\Lambda}f(4\Lambda) = 0$.

If $f'(4\Lambda) = 0$, and $f(4\Lambda) = 0$, it's clear from (10), that GR vacuum solutions, are also vacuum solutions of the $f(R)$ theory, but not all the $f(R)$ vacuum solutions leading to $R = 4\Lambda$, are necessarily GR vacuum solutions, because it's sufficient for any tensor metric $g_{\mu\nu}$ involving $R = 4\Lambda$, to be vacuum solution of the $f(R)$ theory, contrary to Einstein theory, where $R = 4\Lambda$, is a necessary condition but insufficient to imply that $g_{\mu\nu}$ is a vacuum solution.

If we don't take into account the cosmological constant, and we follow the same reasoning, we get the condition $f(0) = 0$, i.e, if $f'(0) \neq 0$, the only vacuum solutions leading to $R = 0$, of the $f(0) = 0$ theories of gravity, are the GR ones, but if $f'(0) = 0$, in addition to GR vacuum solutions, the $f(0) = 0$ modified theories of gravity admit as vacuum solution, any tensor metric leading to $R = 0$.

■

The main conclusion is that the $f(R)$ theories where the $f(R)$ functions are like: (7), (12), (13), and (14) with $f'(0) \neq 0$, are strictly equivalent to general relativity theory, for the same Stress-energy tensor describing the distribution of matter and radiation in a spacetime of constant scalar curvature k , the moment that GR and these modified theories have exactly the same solutions involving $R = k$.

Remark *The conditions(13), (14) with $f'(0) \neq 0$, was also deduced by J. D. Barrow, and A. C. Ottewill[4]. Using a different approach; they showed that given any $f(R)$ theory, if there exists a solution R_0 of $R_0 f'(R_0) = 2f(R_0)$, then the theory contains the GR de Sitter solution with constant curvature R_0 , and with metric scalar factor $a(t)$ identical to the form taken in GR, then when $f(R) = R - \Lambda$, we have simply $R_0 = 4\Lambda$. They also demonstrated that if the energy-momentum tensor is trace-free, then any homogeneous and isotropic solution of GR is also a solution of an $f(R)$ theory provided $f(0) = 0$ and $f'(0) \neq 0$.*

3 construction of the $f(R)$ theories equivalent to GR, when the Ricci scalar is constant

We have proved the conditions on the $f(R)$ function so that every GR solution involving a constant Ricci scalar still to be solution of field's equations generated by this function. With the help of these conditions we are able to have more informations about the form of such functions, in other words starting from any GR solution, it's possible to get an infinity of $f(R)$ theories, were this solution remains solution of field's equations, generated by the $f(R)$ function, when $R = k$, as we will see.

Corollary 4 *For any $g_{\mu\nu}$, solution of Einstein's field equations, such as $g^{\mu\nu}R_{\mu\nu} = k$, $g_{\mu\nu}$ is necessarily solution of field equations generated by the $f(R)$ functions of the form*

$$f(R) = \exp(-R) \left(\int F(R) \exp(R) dR + c_1 \right), \quad (16)$$

$$f(R) = \pm \sqrt{2 \int Q(R) dR + c_2} \text{ with } 2 \int Q(R) dR + c_2 > 0, \quad (17)$$

where $F(R)$, $Q(R)$, are any functions satisfying respectively

$$\begin{aligned} F(k) &= k - 2\Lambda + 1, \\ Q(k) &= k - 2\Lambda. \end{aligned} \quad (18)$$

The constants c_1, c_2 are chosen so that

$$\begin{aligned} \left[\int F(R) \exp(R) dR \right]_{R=k} &= (k - 2\Lambda) \exp(k) - c_1, \\ \left[\int Q(R) dR \right]_{R=k} &= \frac{(k - 2\Lambda)^2 - c_2}{2}, \end{aligned} \quad (19)$$

Proof. Let $g_{\mu\nu}$ be a GR solution, such as $g^{\mu\nu}R_{\mu\nu} = k$. If the function $f(R)$, satisfies the conditions (12), $g_{\mu\nu}$ is automatically solution of $f(R)$ field equations, and we have necessarily.

$$f'(k) + f(k) = k - 2\Lambda + 1. \quad (20)$$

We consider the linear ordinary differential equation

$$f'(R) + f(R) = F(R), \quad (21)$$

were $F(R)$ is any function of R , such as $F(k) = k - 2\Lambda + 1$, hence, all the $f(R)$ functions solutions of (21) satisfy (20) at the value $R = k$, therefore these functions have the form $f(R) = \exp(-R) \left(\int F(R) \exp(R) dR + c_1 \right)$, the moment that this is the general solution of (21), where c_1 is the constant of integration. When we impose to this general solutions, the conditions (12), we get (19). The same reasoning is followed for the functions of the form (17), where we use the fact that

$$f'(k) f(k) = k - 2\Lambda, \quad (22)$$

then $f(R) = \pm \sqrt{2 \int Q(R) dR + c_2}$ is the solutions of

$$f'(R) f(R) = Q(R), \quad (23)$$

with $Q(R)$, satisfying the two conditions (18), (19). ■

Note that (16), (17) are not the only forms of $f(R)$ functions that we can deduce, because the same technique can be used with different combinations of $f'(R)$, and $f(R)$ in order to get different types of ordinary differential equations, and in the same way, we deduce others possible forms of $f(R)$ functions.

Corollary 5 *General relativity vacuum solutions, are also vacuum solutions of every $f(R)$ theory, when the function $f(R)$ has the form*

$$f(R) = \exp\left(\frac{R}{2\Lambda}\right) \left[\int F(R) \exp\left(-\frac{R}{2\Lambda}\right) dR + c' \right], \quad (24)$$

where $F(R)$, is any function, vanishing at $R = 4\Lambda$:

$$F(4\Lambda) = 0, \quad (25)$$

and c' , an arbitrary real constant.

Proof. The proof is exactly the same as the previous one, indeed let $F(R)$ be any function with $F(4\Lambda) = 0$, the ordinary differential equation $f'(R) - \frac{1}{2\Lambda}f(R) = F(R)$ whose general solution is (24), is equivalent to (13) when $R = 4\Lambda$, so the functions (24) will generate field's equations were GR vacuum solutions, are still vacuum solutions. ■

4 Conditions on the parameters of an $f(R)$ model

Another implication of the theorem and its corollaries, is to provide for any class of $f(R)$ functions depending on one or many parameters, conditions on these parameters, so that the $f(R)$ theory admit as solution, a GR solution involving $R = k$, when such condition exists.

For an illustration we consider the model

$$f(R; \alpha, \beta) = R + \alpha \beta^{1-p} R^p, \text{ with: } 0 < p < 1, \alpha, \beta > 0, \quad (26)$$

proposed in[5]as a candidate for dark energy. From (13), (14) we can say that all vacuum solutions of Einstein's field equations, are solution of field equations generated by $R + \alpha\beta^{1-p}R^p$, if and only if

$$\alpha\beta^{1-p} = \frac{4\Lambda}{(p-2)(4\Lambda)^p}, \quad (27)$$

and when we don't take into consideration the cosmological constant, all GR vacuum solutions of Einstein's, are necessarily vacuum solutions of $R + \alpha\beta^{1-p}R^p$ field equations, the moment that $\forall\alpha, \beta; f(0; \alpha, \beta) = 0$. But for this model the sufficient condition(12) which is in fact not a necessary one for the existence of the equivalence GR- $f(R)$ theory when $R = k$, is not satisfied, because it exists no relation between α, β and k , so that (12) is true, and according to (11), we can say that in general, it's necessary that α, β satisfy

$$\alpha\beta^{1-p} = \frac{4\Lambda}{(p-2)k^p}, \quad (28)$$

in order that GR solutions involving $R = k$, remain solutions of field equations generated by this model.

Conclusion

Finding an equivalence between general relativity and $f(R)$ modified theory of gravity, in the meaning that the two theories have exactly the same mathematical solutions for a distribution of matter and radiation described by a stress-energy tensor in a spacetime of constant scalar curvature, doesn't really need a complicated mathematical mechanism, Indeed as we showed, the main idea to achieve this aim, is to bring back Einstein's field equations inside the $f(R)$ ones(5), which provides the necessary and the sufficient condition on the $f(R)$ function(7), so that the equivalence becomes an evidence. The consequence is to obtain certain conditions on the $f(R)$ functions and its first derivative(11), (12), (13), (14), which allows by different manners the obtainment of the $f(R)$ equivalent theories(16), (17), (24), and gives also the possibility to get the conditions on the parameters of any $f(R)$ model, so that this one will be equivalent to GR, when Ricci scalar is a certain constant, if such equivalence exists, as illustrated in (27), (28) for the dark energy model (26), were getting an exact solutions of field equations generated by this model, is a hard task, even for the vacuum case, but with the help of the criteria (14), without being obliged to solve the complicated field equations, we know that GR vacuum solutions, are still vacuum solution of the $f(R)$ theory, which is very convenient and constitute another reason to look for the equivalence between GR and $f(R)$ theory. The study of solutions implying spacetimes with a constant Ricci scalar is not of less importance in general relativity, astrophysics and cosmology, for instance the best description of black holes in the universe, is given by the Schwarzschild and De Sitter solutions, even more the existence of common solutions between general relativity and the modified theory can be considered in the study of concrete physical issue, like the collapse of stars in the formation of a black holes, as proved by Stephen Hawking for the theory of Brans Dicke([6]) were he showed

that a stationary space containing black hole is solution of the Brans-Dicke field equations if and only if it's a solution of Einstein field equations. Globally knowing the relation between GR and any modified theory of gravity is a very good way to know how much the new theory is different from the oldest one, especially when the goal of any modification is to go beyond Einstein theory.

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